Barker College KRB Maths Dept.

2002 TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

PM FRIDAY 16 AUGUST

Staff Involved:

- LJP*
- MRB
- CFR
- AES
- HG
- BHC

90 copies

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 8
- ALL necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged working

Total marks (84)

- Attempt Questions 1 – 7
- All questions are of equal value

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Attempt Questions 1 – 7
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (12 marks) **[BEGIN A NEW PAGE]**

(a) Find the exact value of $\int_0^1 \frac{x}{x^2+1}\,dx$ **2**

(b) Find $\frac{d}{dx}\left(e^{x^2}.\cos^2 x\right)$ **2**

(c) Find the coordinates of the point P which divides A(–3, 8) and B(2, 1) externally in the ratio of 7 : 2. **2**

(d) Use the substitution $x = 5\sin\theta$ to evaluate $\int_{-5}^{5} \frac{dx}{\sqrt{25 - x^2}}$ **3**

(e) Solve: $\frac{3x-2}{x+3} > 1$ **3**

Marks

Question 2 (12 marks) **[BEGIN A NEW PAGE]**

(a) (i) Find $\int_0^{\frac{\pi}{2}} \sin^2 2x \, dx$ 3

(ii) Differentiate $\left(\tan^{-1} x\right)^2$. Hence, evaluate $\int_{-1}^{\sqrt{3}} \frac{\tan^{-1} x}{1 + x^2} dx$ 3

(b) Find the exact value of the coefficient of x^{12} in the expansion $\left(2x - \frac{1}{x^2}\right)^{30}$ 3

(c) Write the expansion of $\sin(A - B)$
Hence, or otherwise, find the exact value of $\sin 15^\circ$ 3

Question 3 (12 marks) **[BEGIN A NEW PAGE]**

(a) $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

(i) State the domain and range 2

(ii) Hence, sketch the graph of $y = f(x)$, clearly showing this information 1

(b) Show that $\tan^{-1}(4) - \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$ 3

(c) If $2\sin^{-1} x = \cos^{-1} x$, find x when $0 \le x \le 1$ 3

(d) Consider the function $f(x) = (x - 1)^2 + 2$

(i) Sketch the graph of $y = f(x)$, showing the coordinates of the vertex. 1

(ii) Find the largest domain for which $f(x)$ has an inverse function $f^{-1}(x)$ 1

(iii) State the domain of $f^{-1}(x)$ 1

Marks

Question 4 (12 marks) **[BEGIN A NEW PAGE]**

(a) The function $f(x) = \ell n(x) - \cos x$ has a zero near $x = 1.2$
Use one application of Newton's Method to find a second approximation for this zero. Write your answer correct to 2 decimal places. 3

(b) The velocity, v m/s, of a particle in Simple Harmonic Motion is given by

$$v^2 = 2x(6 - x)$$

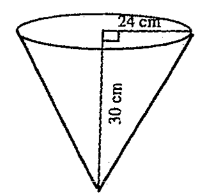
(i) Find the acceleration of the particle. 2

(ii) Prove that the particle always remains in the domain $0 \le x \le 6$ 1

(iii) Find the centre of the motion. 1

(iv) What is the maximum speed of the particle? 1

(c) Water is poured into a conical vessel of height 30 cm and radius of 24 cm.



(i) Show that the volume of water is given by $v = \frac{16\pi h^3}{75}$
when the depth of water is h metres. 1

(ii) If the depth of water is increasing at the rate of $\frac{1}{2}$ cm/min, find the rate of increase of the volume of water when the depth of water is 20 cm. 3

Marks

Question 5 (12 marks) **[BEGIN A NEW PAGE]**

(a) Prove the identity $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$ **2**

(b) (i) Write $\sqrt{3}\sin\theta - \cos\theta$ in the form $R\sin(\theta - \alpha)$ where $R > 0$, and α is acute. **2**

(ii) Find the minimum value of $\sqrt{3}\sin\theta - \cos\theta$ **1**

(iii) Find the general solution of $\sqrt{3}\sin\theta - \cos\theta = \sqrt{3}$ **1**

(c) $P(2ap,\ ap^2)$ is a point on the parabola $4ay = x^2$

(i) Show that the normal to the parabola, at P, has the equation

$$x + py = 2ap + ap^3$$ **2**

(ii) The normal meets the axis of the parabola at L.
Find the coordinates of L. **1**

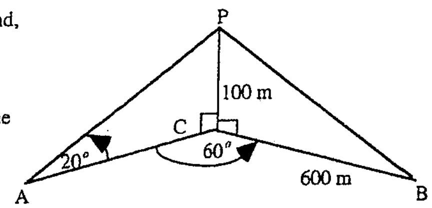
(iii) Find the coordinates of J, the midpoint of LP. **1**

(iv) Show that the locus of J is a parabola and give its vertex. **2**

Marks

Question 6 (12 marks) **[BEGIN A NEW PAGE]**

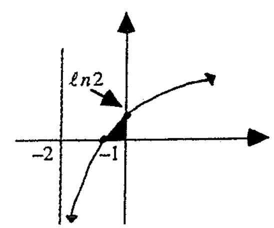
(a) Two stones at A and B, on level ground, subtend an angle of 60° at the base C, of a flagpole.
From A, the angle of elevation to P, the top of the flagpole, is 20°.
B is 600 m from C.
The flagpole CP is 100 m in length.



(i) Find the length of AC. **1**

(ii) Calculate the distance from A to B, to the nearest metre. **2**

(b) The sketch of the curve $y = \log_e(x + 2)$ is shown.



If the shaded area is rotated about the y-axis, find the volume of revolution generated. **3**

(c) (i) By considering the sum of an arithmetic series, show that

$$(1 + 2 + 3 + \dots + n)^2 = \frac{1}{4}n^2(n + 1)^2$$ **2**

(ii) Hence, use the Principle of Mathematic Induction to prove that

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

for $n \geq 1$ **4**

Marks

Question 7 (12 marks) **[BEGIN A NEW PAGE]**

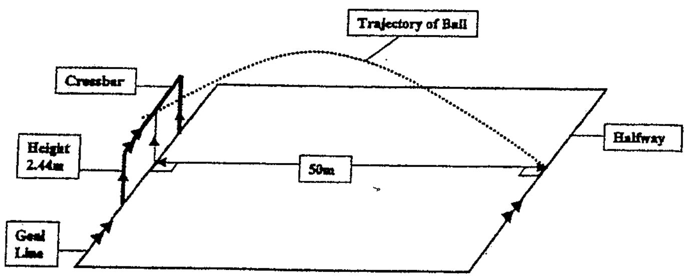
(a) Use the expansion of $(1 + x)^n$ to prove that:

(i) $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n}$ **2**

(ii) $n.2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$ **2**

(b) In the World Cup final, Ros decides to attempt to kick a goal from the half-way line which is 50 m from the "goal" line. She knows that the "best" angle to kick the ball is 45° from the horizontal.

(i) Assuming that there is no force, except gravity, acting on the ball, with $g = 10$ m/s², derive the equations of motion for x and y. **2**



(ii) The crossbar is 2.44 metres high. How fast must she kick the ball in order that it just passes under the crossbar, which is horizontal? (Ignore the size of ball and the thickness of the crossbar). **3**

(iii) How long does the ball take to pass under the crossbar? **1**

(iv) If, instead, Ros had kicked the ball with a velocity of 26 m/s, how high above the crossbar would the ball have passed? **2**

End of Paper

YEAR 12 TRIAL 2002 ANSWERS

1. a) $I = \left[\frac{1}{2}\log(x^2+1)\right]_0^1$

$= \frac{1}{2}\{\log 2 - \log 1\}$

$= \frac{1}{2}\log 2.$

b) $\frac{d}{dx} = e^{x^2}.2\cos x.\sin x + \cos^2 x.e^{x^2}.2x$

$= 2\cos x.e^{x^2}(\sin x + x\cos x)$

c) $P(x, y) = \left(\frac{14+6}{5}, \frac{7-16}{5}\right)$

$= (4, -11/5)$

d) $x = 5\sin\theta$

$\frac{dx}{d\theta} = 5\cos\theta$

$dx = 5\cos\theta\, d\theta$

if $x = 5\sin\theta$

when $x = -5$, $-5 = 5\sin\theta$

$\therefore \theta = -\pi/2$

when $x = 5$, $5 = 5\sin\theta$

$\therefore \theta = \pi/2$

$I = \int_{-\pi/2}^{\pi/2} \frac{5\cos\theta}{5\cos\theta}.d\theta$

$= \int_{-\pi/2}^{\pi/2} d\theta$

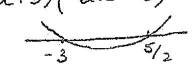
$= [\theta]_{\pi/2}^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

e) $\frac{3x-2}{x+3} > 1 \quad (\times (x+3)^2)$

$(3x-2)(x+3) > (x+3)^2$

$(3x-2)(x+3) - (x+3)^2 > 0$

$(x+3)(2x-5) > 0$



$\therefore x < -3$ OR $x > 5/2.$

2 a) i) $I = \frac{1}{2}\int_0^{\pi/2}(1 - \cos 4x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi/2}$

$= \frac{1}{2}[\pi/2 - 0]$

$= \pi/4.$

ii) $\frac{d}{dx}(\tan^{-1}x)^2 = 2\tan^{-1}x.\frac{1}{1+x^2}$

$\int_{-1}^{\sqrt{3}} \frac{\tan^{-1}x}{1+x^2}\,dx = \left[\frac{1}{2}(\tan^{-1}x)^2\right]_{-1}^{\sqrt{3}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - \left(\frac{-\pi}{4}\right)^2\right]$

$= \frac{1}{2}\left[\frac{\pi^2}{9} - \frac{\pi^2}{16}\right]$

$= \frac{7\pi^2}{288}.$

b) $T_{r+1} = {}^{30}C_r\,(2x)^{30-r}.(-x^{-2})^r$

$= {}^{30}C_r\,2^{30-r}.x^{30-r}.(-1)^r.x^{-2r}$

$= {}^{30}C_r.2^{30-r}.(-1)^r.x^{30-3r}$

let $30 - 3r = 12$

$\therefore r = 6.$

$\therefore T_7 = {}^{30}C_6.2^{24}.(-1)^6.x^{12}$

$\therefore$ coeff. of x^{12} is ${}^{30}C_6.2^{24}.$

c) i) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

ii) $\sin 15° = \sin(45 - 30)$

$= \sin 45.\cos 30 - \cos 45.\sin 30$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}.$

OR $\sin 15° = \sin(60 - 45)$

$= \sin 60\cos 45 - \cos 60\sin 45$

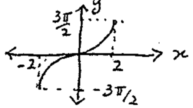
$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}.$

3. a) i) Domain: $-1 \le x/2 \le 1$

$-2 \le x \le 2$

Range: $-3\pi/2 \le y \le 3\pi/2$

ii) 

b) let $a = \tan^{-1}4$, $b = \tan^{-1}3/5$

$\tan a = 4$ $\quad\tan b = 3/5$

$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

$= \frac{4 - 3/5}{1 + 4\times\frac{3}{5}} = 1.$

$\therefore \tan(a-b) = 1$

$a - b = \tan^{-1}1$

$\therefore \tan^{-1}4 - \tan^{-1}3/5 = \pi/4.$

c) $2\sin^{-1}x = \cos^{-1}x$

let $a = \sin^{-1}x \Rightarrow x = \sin a$

$\therefore 2\sin^{-1}x = 2a = \cos^{-1}x$

$\therefore 2a = \cos^{-1}(\sin a)$

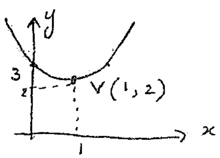
$= \cos^{-1}(\cos(\pi/2 - a))$

$= \pi/2 - a.$

$3a = \pi/2$

$\therefore a = \pi/6.$

$\therefore x = \sin\pi/6 = 1/2.$

d) i) 

ii) $x \ge 1$ OR $x \le 1$.

iii) $x \ge 2$ OR $x \le 2$.

4. a) $f(x) = \ln x - \cos x$

$f'(x) = \frac{1}{x} + \sin x$

$x_1 = 1.2$ $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1{\cdot}2 - \frac{\ln 1.2 - \cos 1.2}{\frac{1}{1.2} + \sin 1.2}$

$\therefore x_2 = 1.3.$

b) i) $v^2 = 12x - 2x^2$

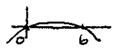
$\frac{1}{2}v^2 = 6x - x^2$

$a = \frac{d}{dx}(\frac{1}{2}v^2) = 6 - 2x$

$\ddot{x} = -2(x-3).$

ii) Since $v^2 = 2x(6-x)$

and $v^2 \ge 0$ then $2x(6-x) \ge 0$



$\therefore 0 \le x \le 6.$

iii) $x = 3.$

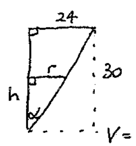
iv) when $\ddot{x} = 0$ $\therefore -2(x-3) = 0$

$\therefore x = 3$

when $x=3$ $\quad v^2 = 2x(6-x)$

$v^2 = 18$

$\therefore v = 3\sqrt{2}$ m/s. is max. speed.

c) i) 

$\frac{r}{h} = \frac{24}{30}$

$\therefore r = \frac{4h}{5}.$

$V = \frac{1}{3}\times\pi r^2\times h$

$= \frac{1}{3}\times\pi\times\frac{16h^2}{25}\times h$

$= \frac{16\pi h^3}{75}.$

ii) $\frac{dV}{dh} = \frac{16}{25}\pi h^2$, $\quad\frac{dh}{dt} = \frac{1}{2}$

$\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$

$= \frac{16}{25}\times\pi\times h^2\times\frac{1}{2}$

When $h = 20$, $\frac{dV}{dt} = \frac{8}{25}\times\pi\times 400$

$= 128\pi$ cm³/min

5. a) LHS $= \frac{2\sin x.\cos x}{2\cos^2 x}$

$= \frac{\sin x}{\cos x}$

$= \tan x$

$=$ RHS.

b) i) $\sqrt{3}\sin\theta - \cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$\therefore R\cos\alpha = \sqrt{3}$, $R\sin\alpha = 1$

$R^2(\cos^2\alpha + \sin^2\alpha) = 3 + 1 = 4$

$\therefore R = 2.$

$\cos\alpha = \sqrt{3}/2$, $\quad\sin\alpha = 1/2$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \pi/6.$

$\sqrt{3}\sin\theta - \cos\theta = 2\sin(\theta - \pi/6).$

ii) min. value is -2.

iii) $2\sin(\theta - \pi/6) = \sqrt{3}$

$\sin(\theta - \pi/6) = \frac{\sqrt{3}}{2}$ $\sin\pi/3$

$\theta - \pi/6 = n\pi + (-1)^n\pi/3.$

~~since sin~~

$\theta = n\pi + \frac{\pi}{6} + (-1)^n\pi/3$

OR $\theta = \frac{(6n+1)\pi}{6} + (-1)^n\frac{\pi}{3}.$

c) i) $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a} = \frac{x}{2a}.$

when $x = 2ap$, $y' = p$.

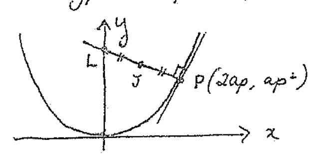
$\therefore$ m of normal $= \frac{-1}{p}$.

Point $(2ap, ap^2)$, $m = \frac{-1}{p}$.

$\therefore y - ap^2 = \frac{-1}{p}(x - 2ap)$

$yp - ap^3 = -x + 2ap$

$\therefore x + yp = 2ap + ap^3.$



ii) L. $x = 0$,

$\therefore yp = 2ap + ap^3$

$y = 2a + ap^2.$

$L(0, 2a + ap^2).$

iii) $J\left(\frac{0 + 2ap}{2}, \frac{2a + ap^2 + ap^2}{2}\right)$

$J(ap, a + ap^2).$

iv) $x = ap$, $\quad y = ap^2 + a$

$p = \frac{x}{a}$ $\quad\therefore y = a.\frac{x^2}{a^2} + a$

$ay = x^2 + a^2$

$x^2 = ay - a^2$

$x^2 = a(y - a)$

$\therefore$ parabola $V(0, a)$ $\quad S = \frac{a}{4}.$

6. a) i) $\tan 20° = \frac{100}{AC}$

$\therefore AC = \frac{100}{\tan 20°} = 274.75$ m (2 d.

ii) $AB^2 = \left(\frac{100}{\tan 20°}\right)^2 + 600^2 - 2.\frac{100}{\tan 20°}.600.\cos 6$

$AB = 520$ m (nearest m).

b) $y = \log_e(x+2) \Rightarrow x + 2 = e^y$

$x = e^y - 2.$

Vol $= \pi\int_0^{\ln 2} x^2\,dy$

$= \pi\int_0^{\ln 2}(e^{2y} - 4e^y + 4)\,dy$

$= \pi\left[\frac{e^{2y}}{2} - 4e^y + 4y\right]_0^{\ln 2}$

$= \pi\left[\left(\frac{1}{2}.e^{2\ln 2} - 4e^{\ln 2} + 4\ln 2\right) - \left(\frac{1}{2}e^0 - 4e^0\right)\right]$

$= \pi\left[\left(\frac{1}{2}\times 4 - 4\times 2 + 4\ln 2\right) - (-3\tfrac{1}{2})\right]$

Vol $= \pi\left[4\ln 2 - 2\tfrac{1}{2}\right]$ units³.

6 c) i) Consider $1+2+3+\dots+n$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_n = \frac{n}{2}\left[2 + (n-1)\right]$$

$$= \frac{n}{2}(n+1)$$

$$\therefore (1+2+3+\dots+n)^2 = \left[\frac{n}{2}(n+1)\right]^2$$

$$= \frac{1}{4}.n^2(n+1)^2.$$

ii) let $n=1$

LHS $= 1^3 = 1$ RHS $= 1^2 = 1$.

$\therefore$ true for $n=1$.

Assume true for $n=k$.

ie $1^3+2^3+3^3+\dots+k^3 = (1+2+\dots+k)^2$

Prove true for $n=k+1$

ie $1^3+2^3+3^3+\dots+(k+1)^3 = (1+2+\dots+(k+1))^2$

PROOF: $S_{k+1} = S_k + T_{k+1}$

$$S_{k+1} = (1+2+\dots+k)^2 + (k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left[k^2+4k+4\right]$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$\therefore$ If true for $n=k$, it is true for $n=k+1$.

Since true for $n=1$, $\therefore$ true for $n=2, 3, \dots$.

$\therefore$ By mathematical induction it is true for $n \geq 1$.

7. a)

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

i) let $x=1$

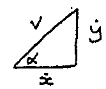
$$\therefore 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

ii) differentiate both sides.

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + n\binom{n}{n}x^{n-1}$$

let $x=1$

b) i) $\ddot{x} = 0$



$\sin\alpha = \frac{\dot{y}}{V}$

$\therefore \dot{y} = V\sin\alpha$

$\cos\alpha = \frac{\dot{x}}{V}$

$\therefore \dot{x} = V\cos\alpha$

$\dot{x} = c_1$

when $t=0$,

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c_2$

when $t=0$, $x=0$

$\therefore x = Vt\cos\alpha$

$\ddot{y} = -10$

$\dot{y} = -10t + c_3$

when $t=0$, $\dot{y} = V\sin\alpha$

$\therefore \dot{y} = -10t + V\sin\alpha$

$y = -5t^2 + Vt\sin\alpha + c_4$

when $t=0$, $y=0$

$\therefore y = -5t^2 + Vt\sin\alpha$.

ii) $\alpha = 45°$,

$\therefore x = Vt\cos 45°$ $y = -5t^2 + Vt\sin 45°$

$x = \frac{Vt}{\sqrt{2}}$ $y = -5t^2 + \frac{Vt}{\sqrt{2}}$

$\therefore t = \frac{\sqrt{2}x}{V}$

to pass under post $x = 50$

$\therefore t = \frac{50\sqrt{2}}{V}$.

Subst. into y:

$$y = -5\left(\frac{50\sqrt{2}}{V}\right)^2 + V.\frac{50\sqrt{2}}{V\sqrt{2}}$$

$$y = \frac{-25000}{V^2} + 50$$

height of post is $y = 2.44$

$$\therefore 2.44 = \frac{-25000}{V^2} + 50$$

$$V^2 = \frac{25000}{47.56} = 525.652$$

$\therefore V = 22.93$ m/s.

7. iii) $t = \frac{50\sqrt{2}}{V} = \frac{50\sqrt{2}}{22.927\dots}$

$\therefore t = 3.084$ sec.

iv) $V = 26$, $x = 50$

$$y = -5t^2 + \frac{Vt}{\sqrt{2}}$$

But $t = \frac{\sqrt{2}x}{V} = \frac{\sqrt{2}\times 50}{26}$

$$\therefore y = -5\left(\frac{50\sqrt{2}}{26}\right)^2 + \frac{26}{\sqrt{2}}.\frac{50\sqrt{2}}{26}$$

$= 13.02$ m

$\therefore$ it passes

$13.02 - 2.44 = 10.58$

10.58 m above the post.